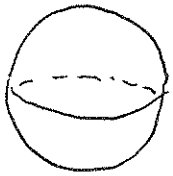
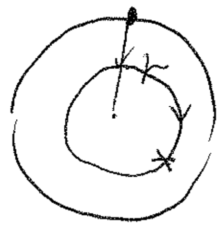
Chapter 10. Parametric Equations and Polar Coordinates.

Motivation: Cartesian coordinate system VS other system.





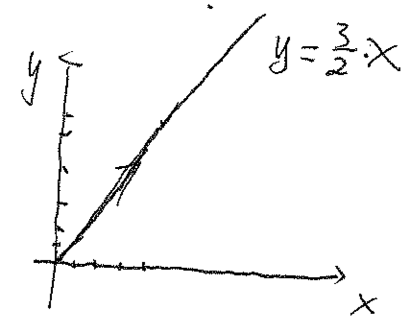
$x^2+y^2=r^2$

Parametric Equations: which parameter t (time)

e.g. A car is moving on the XY plane starting from origin. Its velocity along x-direction is 2 (m/s) and along y-direction 3 (m/s)

Describe the position of the car after t second.

position \ time	$t=0$	$t=1$	$t=2$	....	t
x	0	2	4	.....	$x=2t$
y	0	3	6	.....	$y=3t$



The equations $\begin{cases} x=2t \\ y=3t \end{cases}$ tell us more than (in XY system, we cannot see the time variable t)

the the line (equation) $y=\frac{3}{2}x$ does.

Cartesian equation   Parametric equations

$$y=\frac{3}{2}\cdot x \longleftrightarrow \begin{cases} x=2t \\ y=3t \end{cases}$$

§10.1 Curves Defined by Parametric Equations.

In general, if x, y are both given as ~~a~~ functions of a third variable t, by

$x = f(t)$, $y = g(t)$, then we the point $(x, y) = (f(t), g(t))$ determines a curve C, which we call a $\boxed{\text{parametric curve.}}$

$x = f(t)$, $y = g(t)$ are called $\boxed{\text{parametric equations}}$ of the curve

$\boxed{\text{Goals for chapt 10.}}$

- for Given parametric equations, sketch the curve / describe the motion.
- Change parametric equations to Cartesian equation
- Give Cartesian equation, find proper parametric equations.
- One particular parametric equations system.
- Calculus (derivative / integral) related to parametric equations.

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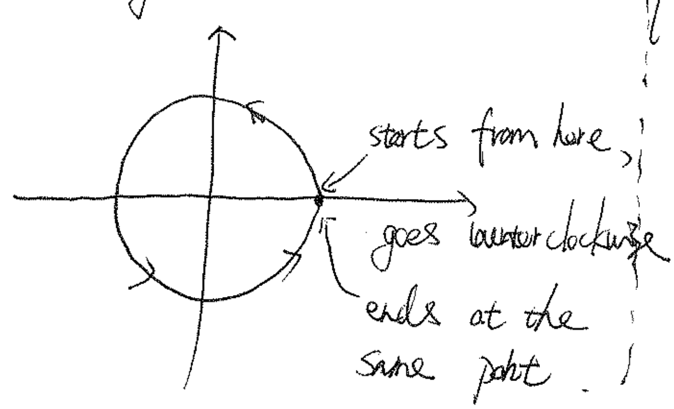
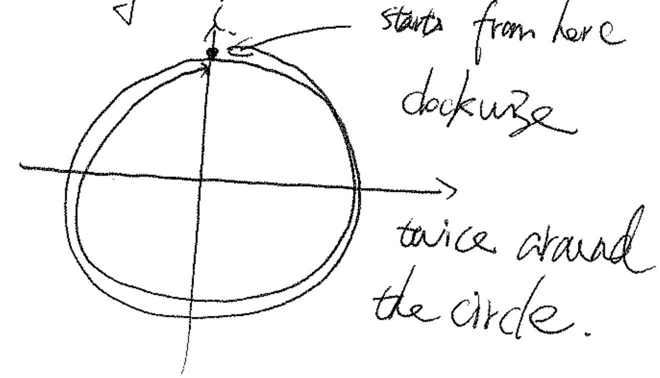
eg. What curves are represented by the following parametric equations

$x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$ | $x = \sin(2t)$, $y = \cos(2t)$, $0 \le t \le 2\pi$.

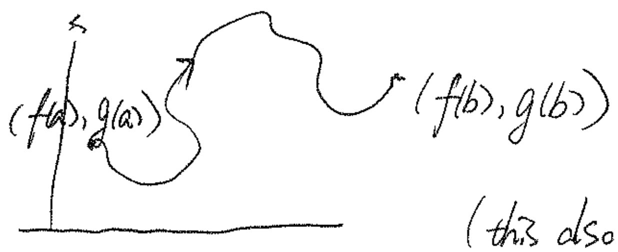
(Hint: eliminate t from both equations via trig-identity)

$x^2 + y^2 = 1$ <— same Cartesian equation —> $x^2 + y^2 = 1$.





Remark: the interval $[a,b]$ for t (ie. $t\in[a,b]$) is important for parametric curve. It tells us where to start and where to end. Actually, for $x=f(t)$, $y=g(t)$, $t\in[a,b]$,



the curve starts from $(f(a), g(a))$ and ends at $(f(b), g(b))$

(this also tells us the "direction" of the motion)

- Parametric equations for line segments.

eg. Sketch the curve and find the Cartesian eq for the following parametric eqns.

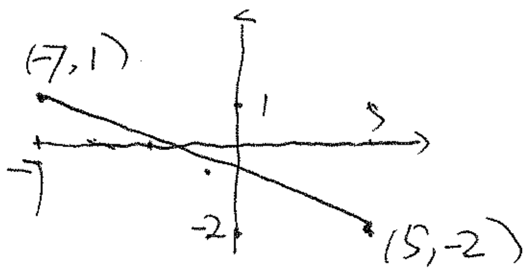
$$\begin{cases} x = 1-2t \\ y=\frac{1}{2}t-1 \end{cases} \quad -2\le t\le 4.$$

$t=-2$ $x=5$, $y=-2$

$t=4$ $x=-7$, $y=1$

eliminate we have $t=\frac{1-x}{2}$

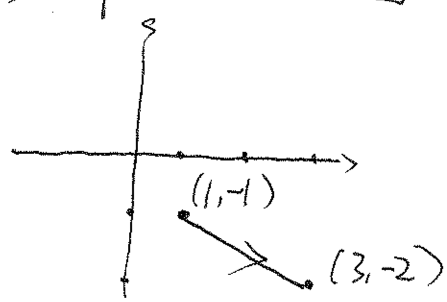
$y=\frac{1}{2}\cdot\frac{1-x}{2}-1=-\frac{1}{4}x-\frac{3}{4}$.



We are particularly interested in $t\in[0,1]$ sometimes.

eg. Find a parametrization of the line segment starting at $(x,y)=(1,-1)$ and ending at $(x,y)=(3,-2)$ for $t\in[0,1]$

$$\begin{cases} x=\underline{1}+\underline{2}\,t \\ y=\underline{-1}+\underline{-1}\,t \end{cases}$$



More examples

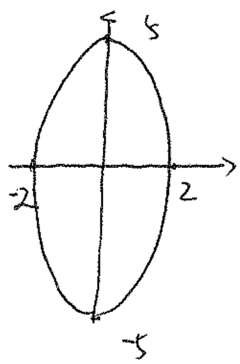
e.g. Ellipse : A curve is prepresented parametrically by

$$x = -2\cos 3t \ , \ y = 5\cdot\sin 3t \ , \quad t \in [0, \tfrac{\pi}{6}]$$

sketch the graph (indicate the direction of the motion) and find its Cartesian equation.

sln: Eliminate t via trig-identity: $\cos^2\square + \sin^2\square = 1$

$$\left(\frac{x}{-2}\right)^2 + \left(\frac{y}{5}\right)^2 = (\cos 3t)^2 + (\sin 3t)^2 = 1 \quad \text{ie.} \quad \boxed{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{5}\right)^2 = 1}$$



Starts at $t=0$

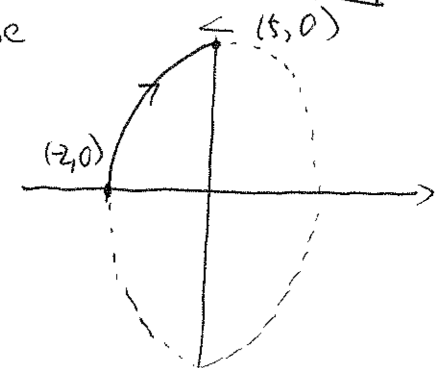
$x=-2$, $y=0$

Ends at $t=\frac{\pi}{6}$

$x=-2\cdot\cos\frac{\pi}{2}=0$, $y=5\sin\frac{\pi}{2}=5$

$\frac{1}{4}$-ellipse

clockwise.

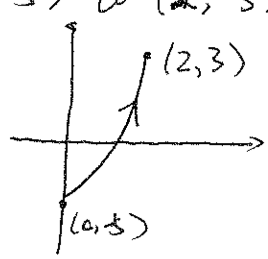


Remark: the range for y-variable is $[0,5]$ ie. $0 \le y \le 5$

e.g. One 'trivial' parametric equations.

curve: $y = 2x^2 - 5$. from $(0,-5)$ to $(2,3)$ can be parameterized via

$$\begin{cases} x = t \\ y = 2t^2 - 5 \end{cases} \quad t \in [0, 2]$$



Remark: In general, any function $y=f(x)$ can be parameterized as $\begin{cases} x = t \\ y = f(t) \end{cases}$.

Hint for ww7: Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The natural way to parametrize it is via $x = a\cdot\cos t$, $y = b\sin t$ (or $x = \pm a\cos t$, $y = \pm b\sin t$)

In ww7, since it is the bottom part, you need to consider $y = -b\cdot\sin t$.

§10.2 Calculus with Parametric Curves

(Differentiation)

- Tangents: Give parametric equations $x = f(t)$, $y = g(t)$.

Then $$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \text{ i.e. } y'(x) = \frac{g'(t)}{f'(t)}}$$ (in the formula sheet)

Remark: the above formula follows from Chain Rule. $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

for $y = y(x) = y \circ x(t)$

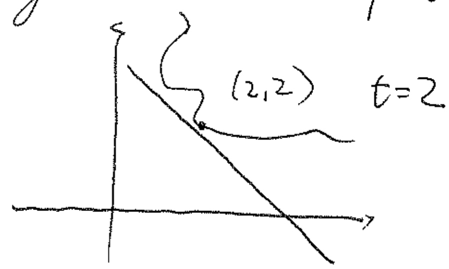
e.g. 1. Consider $x = 6 - t^2$, $y = t^3 - 3t$. Find the derivative of y with respect to x as a function of t.

$$y'(x) = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2-3}{-2t}$$

- The most important application for the above formula is to EVALUATE the derivative at some specific point, i.e. find the SLOPE of the tangent line at this point and find the FORMULA of the tangent line of the curve.

Remark: Tangent line of $y = y(x)$ at (x_0, y_0): $y = y_0 + y'(x_0) \cdot (x - x_0)$

e.g. 2. Consider the parametric equations in e.g. 1. What are the coordinates of the curve at $t=2$? What's the slope of the tangent line at that point? Find the tangent line.



$t=2$ $\quad x = 6 - 2^2 = 2$, $y = 2^3 - 3 \cdot 2 = 2$.

Slope: $\frac{dy}{dx} = \frac{3t^2-3}{-2t} = \frac{3 \cdot 4 - 3}{-4} = -\frac{9}{4}$

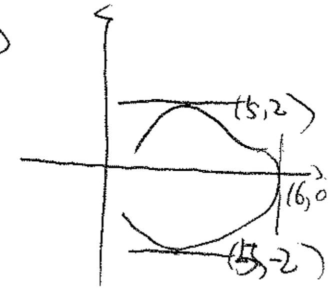
tangent line: $\boxed{y = 2 - \frac{9}{4} \cdot (x-2) = -\frac{9}{4} x + \frac{13}{2}}$

e.g. 3. Find all the points where the curve has horizontal tangent line in eg 1,2.

Hint: horizontal tangent line $\iff \frac{dy}{dx} = 0 \iff \frac{dy}{dt} = 0$

i.e. $y'(x) = \frac{3t^2-3}{-2t} = 0 \Rightarrow 3t^2-3=0 \Rightarrow t^2=1 \Rightarrow t=1$ or $t=-1$.

$t=1$, $(x,y)=(5,-2)$. $t=-1$, $(x,y)=(5,2)$



e.g. 4 Points with vertical tangent line.

Hint: Vertical tangent line: $\frac{dy}{dx} = \infty \iff \frac{dx}{dt} = 0$.

$y'(x) = \frac{3t^2-3}{-2t} = \infty \Rightarrow -2t=0 \Rightarrow t=0 \Rightarrow (x,y)=(6,0)$

Remark: Vertical tangent line can also be viewed as $\frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}} = 0$.

e.g. 5. Implicit differential rule (Hint for ww 6).

The curve is defined IMPLICITLY via the following parametric equations.

$x^3 - 2 t^2 = 7$, $2y^3 + t = 18$. Find the slope of the tangent line at $t=2$.

Solution: $t=2$, want to compute $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ at $t=2$, $(x,y)=(1,2)$

Take derivate w.r.t. t IMPLICITLY in both equations.

$\frac{d}{dt}(x^3-2t^2) = \frac{d}{dt}(7) \Rightarrow 3\cdot x^2 \cdot \boxed{\frac{dx}{dt}} - 4\cdot t = 0$

$\Rightarrow \frac{dx}{dt} = \frac{4t}{3x^2} = \frac{4\cdot 2}{3\cdot 1} = \frac{8}{3}$

$\frac{d}{dt}(2y^3+t) = 0 \Rightarrow 2\cdot 3y^2 \cdot \boxed{\frac{dy}{dt}} + 1 = 0 \Rightarrow \frac{dy}{dt} = -\frac{1}{6y^2} = -\frac{1}{24}$

$\Rightarrow \frac{dy}{dx} = \frac{-\frac{1}{24}}{\frac{8}{3}} = \boxed{-\frac{1}{64}}$

☆ For $(x, y) = (f(t), g(t))$, the tangent line can also be represented parametrically at $t=a$, the parametric formula for the tangent line is

$$\begin{cases} x = f(a) + f'(a)\, t \\ y = g(a) + g'(a)\, t \end{cases}$$

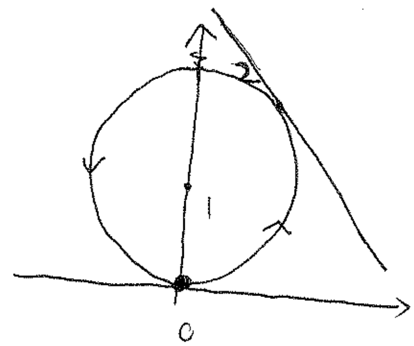
☆ eg 6. (Final 14). Consider the parametric curve given by $x = \cos t$, $y = 1 + \sin t$, $t \in [0, 2\pi)$. (a). Give the sketch of the curve
(b) Give the parametric formula for the tangent line at $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$

Solution: (a). Cartesian equation: $x^2 + (y-1)^2 = 1$.

A circle of radius 1 centered at $(0, 1)$

counter-clockwise

Starting and ending at the same point $(0, 0)$



☆ (b). Find t value for the given point: $\frac{\sqrt{3}}{2} = \cos t$, $\frac{3}{2} = 1 + \sin t \Leftrightarrow \frac{1}{2} = \sin t$

$$\Rightarrow \boxed{t = \frac{\pi}{6}}$$

Compute $f'(t), g'(t)$ at $t = \frac{\pi}{6}$

$$\frac{dx}{dt} = x'(t) = (\cos t)' = -\sin t = -\sin\frac{\pi}{6} = -\frac{1}{2}.$$

$$\frac{dy}{dt} = y'(t) = (1+\sin t)' = \cos t = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

Therefore, the parametric tangent line at $t = \frac{\pi}{6}$ is

$(x, y) = \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$

$$\boxed{\begin{cases} x = \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot t \\ y = \frac{3}{2} + \frac{\sqrt{3}}{2} \cdot t. \end{cases}}$$ ☆

$t \in (-\infty, \infty)$

Remark: the Cartesian tangent line of (b): $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$.

$y = \frac{3}{2} - \sqrt{3} \cdot \left(x - \frac{\sqrt{3}}{2}\right) \Rightarrow y = -\sqrt{3}x + 3$ can be parameterized as

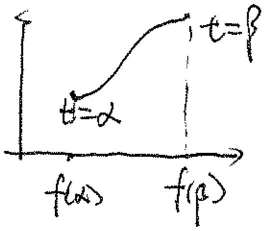
$\begin{cases} x = t \\ y = -\sqrt{3}t + 3 \end{cases}$ which is also ok. (equivalent to Answer ☆)

- (Integration). Arc-length. (Area will be discussed later in §10.4)

If a curve C is described by the parametric equations $x=f(t)$, $y=g(t)$, $\alpha \le t \le \beta$, then the length of C is

$$\star \quad \boxed{L=\int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt} = \int_\alpha^\beta \sqrt{[f'(t)]^2+[g'(t)]^2}\cdot dt$$

Remark: the above formula can be derived from the previous Arc-L. formula and u-sub.



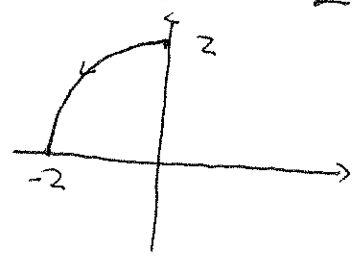
$$L=\int_{f(\alpha)}^{f(\beta)} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx \;\; \overline{\overline{\;x=f(t)\;}} \;\; \int_\alpha^\beta \sqrt{1+\frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\cdot\frac{dx}{dt}\cdot dt$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$dx=\frac{dx}{dt}\cdot dt \qquad =\int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt$$

e.g. Curve $x=2\cos t$, $y=2\sin t$. Compute the arc-length from $t=\frac{\pi}{2}$ to $t=\frac{3\pi}{2}$.

$\frac{dx}{dt}=-2\sin t$, $\frac{dy}{dt}=2\cdot\cos t$.

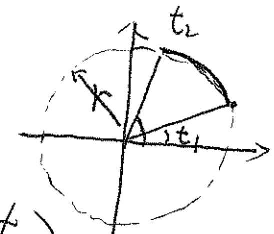


$$\text{Arc-length}=\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{(-2\sin t)^2+(2\cos t)^2}\,dt=\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{4}\,dt$$

$$=2\cdot\left(\frac{3\pi}{2}-\frac{\pi}{2}\right)=\pi.$$

Remark: the arclength of a quarter circle is not easy to compute via Cartesian equation $y=\sqrt{2-x^2}$, $-2\le x\le 0$. via $L=\int_{-2}^{0}\sqrt{1+(y')^2}\cdot dx$.

And in general, above computation works for any segment of the circle



$$L=r\cdot(t_2-t_1)$$

Hint for ww7: Use double angle formula $\cos^2\square=\frac{1+\cos 2\square}{2}$ to simplify the integrand.

Actually, $\sqrt{15^2\cdot 2+15^2\cdot 2\cdot\cos 5t}=\sqrt{15^2\cdot 2\cdot(1+\cos 5t)}=\sqrt{15^2\cdot 2\cdot 2\cdot\cos^2\left(\frac{5t}{2}\right)}=15\cdot 2\cdot\cos\left(\frac{5t}{2}\right)$.